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ON THE POSSIBILITY OF RESONANCE STABILIZATION OF A SYSTEM OF OSCILLATORS*

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It is shown that resonance stabilization is possible in the case of interaction between two unstable oscillators with equal frequencies, i.e. when a resonance is "switched-on" the instability is superseded by asymptotic stability. This effect cannot occur for no other relation between frequencies.

1. Statement of the problem.

We consider the problem of equilibrium position of a system of two linear oscillators linked by the strong nonlinear relationship

$$x_1^{\cdot\cdot} + \omega_1^2 x_1 = f_1(x_1, x_1^{\cdot}) + g_1(x_1, x_2, x_1^{\cdot}, x_2^{\cdot}), \quad x_2^{\cdot\cdot} + \omega_2^2 x_2 = f_2(x_2, x_2^{\cdot}) + g_2(x_1, x_2, x_1^{\cdot}, x_2^{\cdot}) \tag{1.1}$$

$$f_j(0) = g_j(0) = 0, \quad j = 1, 2$$

where the expansion in Taylor series of functions f and g begins with quadratic terms, and function g_j contains only cross terms.

It was shown in /1,2/ that in the absence of resonance between frequencies ω_1 and ω_2 the equilibrium position of system (1.1) can be asymptotically stable only when each of the oscillators is stable. Indeed, the shortened standard form of system (1.1) is

$$z_j^{\cdot} = i\omega_j z_j + z_j[A_{j1}|z_1|^2 + A_{j2}|z_2|^2] \tag{1.2}$$

and the asymptotic stability criterion stipulates the simultaneous fulfillment of the following three conditions:

$$a_{11} = \operatorname{Re} A_{11} < 0, \quad a_{22} = \operatorname{Re} A_{22} < 0 \tag{1.3}$$

$$\Delta = a_{11}a_{22} - a_{12}a_{21} > 0 \text{ for } a_{12} > 0, \ a_{21} > 0, \ a_{ij} = \operatorname{Re} A_{ij}$$

of which the first two ensure the asymptotic stability of each oscillator.

Let us consider the question whether the equilibrium position of system (1.1) is possible when one or both oscillators are unstable. The above exposition implies that this is only possible when there is resonance of frequencies. Since a system with resonance of the fourth or higher order is asymptotically stable or unstable simultaneously with system (1.2) (see /3/), "resonance stabilization" is possible, if at all, only when the frequency ratio is 1:3 ($\omega_2 = 3\omega_1$) or 1:1 ($\omega_2 = \omega_1$) (note that when this ratio is 1:2 ($\omega_2 = 2\omega_1$), the equilibrium position is unstable /4,5/).

It was previously shown (**) that, when the first of conditions (1.3) is violated, a partial resonance stabilization of system (1.1) is possible. Namely, although system (1.2) is unstable when $a_{11} > 0$, the complete resonance system (1.1) may become asymptotically stable when resonance 1:3 is "switched on". A fast oscillator stabilizes the instability generated by the slow oscillator. The necessary condition for asymptotic stability of the considered resonance system is $a_{22} < 0$.

The question of feasibility of resonance stabilization of system (1.1) when each oscillator is unstable, remained open. It will be shown in Sect.2 that such stabilization is only possible when the frequency ratio is 1:1.

In Sects.3 and 4 the stability problem is considered from a more general point of view. Let matrix A of the fourth order linearized system

$$z^{\cdot} = F(z), \quad F(0) = 0 \tag{1.4}$$

have two pairs of pure imaginary eigenvalues

$$\lambda_{1,2} = \pm i\omega_1, \quad \lambda_{3,4} = \pm i\omega_2; \quad \omega_2 = \omega_1$$

Under such conditions the Jordan form of matrix A, as a rule, contains the cell

$$\begin{Vmatrix} i\omega & 1 \\ 0 & -i\omega \end{Vmatrix}$$

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* Prikl.Matem.Mekhan.,44,No.4,660-666,1980

**) Khazin, L. G. and Shpol', E. E., Investigation of asymptotic stability of resonance 1:3. Preprint No.67, Inst. Appl. Math., Akad.Nauk SSSR, 1978.

The problem is of codimension $\nu = 3$ and the equilibrium position is unstable(*). If matrix A is diagonal $(\nu = 4)$ the stability problem cannot be algebraically solved,(**) consequently, simple, readily checked necessary as well as sufficient stability conditions become important. Such conditions are derived in Sects.3 and 4 of the present paper.

2. Two examples of resonance stabilization. Let us consider two oscillators of equal frequencies $\omega_2 = \omega_1 = \omega$ linked by the strong nonlinear relation

$$z_1^{\cdot} = i\omega z_1 + z_1 [A_{11} \mid z_1 \mid^2 + A_{12} \mid z_2 \mid^2] + B_1 \mid z_1 \mid^2 z_2 + B_2 z_1^2 \bar{z}_2 + B_3 z_2 \mid z_2 \mid^2 + B_4 \bar{z}_1 z_2^2 \tag{2.1}$$

$$z_2^{\cdot} = i\omega z_2 + z_2 [A_{21} \mid z_1 \mid^2 + A_{22} \mid z_2 \mid^2] + B_5 z_1 \mid z_1 \mid^2 + B_6 z_1 \mid z_2 \mid^2 + B_7 z_1^2 \bar{z}_2 + B_8 \bar{z}_1 z_2^2$$

where A_{jk} and B_j are complex numbers $A_{jk} = a_{jk} + ic_{jk}$ and $B_j = b_j e^{i\psi_j}$, and $b_j \geqslant 0$. System (2.1) must be supplemented by two conjugate equations.

1°. Let us show that the equilibrium position of (2.1) can be asymptotically stable, even when $a_{11} > 0$ and $a_{22} > 0$ on the following problem of two symmetrically linked oscillators. We select the coefficients of system (2.1) of the form

$$A_{11} = A_{22} = 1;\ A_{12} = A_{21} = -91 / 3;\ B_1 = B_6 = 173 / 3,\ B_2 = B_8 = 1;\ B_3 = B_5 = -42;\ B_4 = B_7 = 0 \tag{2.2}$$

Function

$$L = \mid z_1 \mid^2 + \mid z_2 \mid^2 + {}^1/_2 (\bar{z}_1 z_2 + z_1 \bar{z}_2) \tag{2.3}$$

is the Liapunov function of system (2.1), (2.2).

Indeed, setting $z_k = \sqrt{\rho_k} e^{i\varphi_k},\ \psi = \varphi_2 - \varphi_1$ we obtain

$$L = \rho_1 + \rho_2 + \operatorname{Re} (\sqrt{\rho_1 \rho_2} e^{-i\psi}) \geqslant (\sqrt{\rho_1} - \sqrt{\rho_2} / 2)^2 + {}^3/_4 \rho_2 > 0$$

By virtue of (2.1) and (2.2) we have for dL / dt

$${}^1/_2 L^{\cdot} = -20\ (\rho_1^2 + \rho_2^2) - 2\rho_1\rho_2 + 2 (\rho_1 + \rho_2) \sqrt{\rho_1\rho_2} \cos \psi \leqslant -19 (\rho_1^2 + \rho_2^2) < 0$$

Hence the equilibrium position of system (2.1), (2.2) is asymptotically stable.

Remarks. a) A more general reasoning which yields as a corollary the considered here Liapunov function appears in Sect.3; b) (2.3) is a Liapunov function for the complete system. The asymptotic stability of the complete system also follows from the general theorems on homogeneous systems /6/.

2°. The nonresonance link may prove to be such that system (1.2) is unstable when each of the oscillators is stable, i.e. when $a_{11} < 0$, $a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} < 0$ for $a_{12} > 0$ and $a_{21} > 0$.

We shall prove this on an example that in this case with $\omega_2 = \omega_1$ it is possible to stabilize the system

$$z_1^{\cdot} = iz_1 - 716\, z_1 \mid z_1 \mid^2 + 11 z_1 \mid z_2 \mid^2 - 673 \mid z_1 \mid^2 z_2 - \bar{z}_1 z_2^2$$
$$z_2^{\cdot} = iz_2 + 651\, z_2 \mid z_1 \mid^2 - 10 z_2 \mid z_2 \mid^2 - 1412\ z_1 \mid z_1 \mid^2 \tag{2.4}$$

for which the Liapunov function is

$$L = 2 \mid z_1 \mid^2 + \mid z_2 \mid^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 \geqslant \rho_1 + (\sqrt{\rho_1} - \sqrt{\rho_2})^2 > 0$$

$${}^1/_2 L^{\cdot} \leqslant -20\rho_1^2 - 10\rho_2^2 - 2\rho_1\rho_2 \cos 2\psi + \rho_1 \sqrt{\rho_1\rho_2} \cos \psi \leqslant -{}^1/_2 (39\, \rho_1^2 + 20\rho_2^2 - 5\, \rho_1\rho_2) < 0$$

3. The necessary conditions of stability. The stability of system (2.1) is equivalent to the stability of the following third order system derived from (2.1) by passing to polar coordinates $\rho_k, \varphi_k\ (k = 1, 2)$ using formulas $z_k = \sqrt{\rho_k} e^{i\varphi_k};\ \psi = \varphi_2 - \varphi_1,\ \rho_k \geqslant 0$:

$$\rho_k^{\cdot} = 2 [a_{kk}\rho_k^2 + \Omega_k \rho_1 \rho_2 + \sqrt{\rho_1\rho_2} (\rho_1 \Phi_k + \rho_2 \Psi_k)] \tag{3.1}$$
$$\psi^{\cdot} = P\rho_1 + R\rho_2 + S \sqrt{\rho_1\rho_2} +$$
$$\rho_1^{-1/2} \rho_2^{1/2} [-b_5 \rho_1^2 \sin (\psi - \psi_5) - b_3 \rho_2^2 \sin (\psi + \psi_3)]$$
$$\Omega_1 = a_{12} + b_4 \cos (2\psi + \psi_4);\ \Omega_2 = a_{21} + b_7 \cos (2\psi - \psi_7)$$
$$\Phi_1 = b_1 \cos (\psi + \psi_1) + b_2 \cos (\psi - \psi_2);\ \Phi_2 = b_5 \cos (\psi - \psi_5)$$
$$\Psi_1 = b_3 \cos (\psi + \psi_3),\ \Psi_2 = b_6 \cos (\psi - \psi_6) + b_8 \cos (\psi + \psi_8)$$
$$P = c_{21} - c_{11} - b_7 \sin (\psi - \psi_7)$$
$$R = c_{22} - c_{12} - b_4 \sin (2\psi + \psi_4)$$
$$S = -b_1 \sin (\psi + \psi_1) + b_2 \sin (\psi + \psi_2) - b_6 \sin (\psi - \psi_6) + b_8 \sin (\psi + \psi_8)$$

*) Khazin, L. G., On the resonance instability of equilibrium position at multiple resonance. Preprint No. 97, Inst. Appl. Math., Akad. Nauk SSSR, 1975.

**) Khazina, G. G. and Khazin, L. G., The nonexistence of an algebraic criterion of asymptotic stability at resonance. Preprint No.112, Inst. Appl. Math. Akad. Nauk SSSR, 1977.

Then using the homogeneity of the obtained system with respect to ρ_1 and ρ_2, and introducing coordinates R and $\theta\,(0 \leqslant R < \infty,\ 0 \leqslant \theta \leqslant \pi/2)$ by formulas $\rho_1 = R\cos\theta$, $\rho_2 = R\sin\theta$, and $d\tau = R dt$, we obtain

$$d\ln R / d\tau = 2\Pi(\theta,\psi) = 2\Pi_1(\theta) + 2\Pi_2(\theta,\psi) \tag{3.2}$$

$$d\theta / d\tau = g(\theta,\psi) = g_1(\theta) + g_2(\theta,\psi),\quad d\psi / d\tau = f(\theta,\psi) = f_1(\theta) + f_2(\theta,\psi)$$

$$\Pi_1 = a_{11}\cos^3\theta + a_{12}\cos^2\theta\sin\theta + a_{21}\cos\theta\sin^2\theta + a_{22}\sin^3\theta$$

$$\Pi_2 = \cos^{1/2}\theta\sin^{1/2}\theta\,[\Pi_{11}\cos^2\theta + \Pi_{12}\sin^2\theta + \Pi_{13}\cos\theta\sin\theta) + \cos\theta\sin\theta\,[b_4\cos\theta\cos(2\psi+\psi_4) + b_7\sin\theta\cos(2\psi-\psi_7)]$$

$$\Pi_{11} = b_1\cos(\psi+\psi_1) + b_2\cos(\psi-\psi_2),\ \Pi_{12} = b_6\cos(\psi-\psi_6) + b_8\cos(\psi+\psi_8),\ \Pi_{13} = b_3\cos(\psi+\psi_3) + b_5\cos(\psi-\psi_5)$$

$$g_1 = \cos\theta\sin\theta\,[(a_{21}-a_{11})\cos\theta + (a_{22}-a_{12})\sin\theta]$$

$$g_2 = \cos\theta\sin\theta\,[b_7\cos\theta\cos(2\psi-\psi_7) - b_4\sin\theta\cos(2\psi+\psi_4)] + \cos^{1/2}\theta\sin^{1/2}\theta\,[b_5\cos^2\theta\cos(\psi-\psi_5) + \sin\theta\cos\theta g_{11} - b_3\sin^2\theta\cos(\psi+\psi_3)]$$

$$g_{11} = b_6\cos(\psi-\psi_6) + b_8\cos(\psi+\psi_8) - b_1\cos(\psi+\psi_1) - b_2\cos(\psi-\psi_2)$$

$$f_1 = (c_{21}-c_{11})\cos\theta + (c_{22}-c_{12})\sin\theta$$

$$f_2 = -b_7\cos\theta\sin(2\psi-\psi_7) - b_4\sin\theta\sin(2\psi+\psi_4) - \sin^{-1/2}\theta\cos^{-1/2}\theta\,[b_5\cos^2\theta\sin(\psi-\psi_5) + f_{11}\sin\theta\cos\theta - b_3\sin^2\theta\sin(\psi+\psi_3)]$$

$$f_{11} = -b_1\sin(\psi+\psi_1) + b_2\sin(\psi+\psi_2) - b_6(\psi-\psi_6) + b_8\sin(\psi+\psi_8)$$

The equations in terms of angles in the equations for θ and ψ constitute an independent subsystem, hence, when $\theta(\tau)$, $\psi(\tau)$ is its solution, then

$$R(\tau) = R_0 e^{p(\tau)};\quad p(\tau) = \int_0^\tau \Pi[\theta(\xi), \psi(\xi)]\,d\xi \tag{3.3}$$

Theorem. Let θ_k and ψ_k be nondegenerate solutions of system $f(\theta,\psi) = g(\theta,\psi) = 0$ that is algebraic in $\sin\theta$ and $\cos\theta$. Then the necessary condition the asymptotic stability of system (3.1) for all k is that $\Pi(\theta_k,\psi_k) < 0$.

The proof of this directly follows from formula (3.3).

Corollary. Even if only at one stationary point (θ_k, ψ_k), we have $\Pi(\theta_k,\psi_k) > 0$, the equilibrium position of system (3.1) is unstable.

Remark. Instability of the complete system instability (when the conditions of the theorem are satisfied) is implied by the existence of Chataev's function in the neighborhood of the growing solution (*).

4. The sufficient conditions of stability.

Sufficient conditions of stability can be obtained by requiring that some homogeneous polynomial be a Liapunov function. Even the examination of the simplest second order polynomial as a possible Liapunov function yields nontrivial sufficient conditions, which were used in Sect.2.

Lemma. If system (2.1) admits a Liapunov function of the form

$$L = k|z_1|^2 + |z_2|^2 + a_{11}z_1^2 + \bar a_{11}\bar z_1^2 + a_{12}z_1z_2 + \bar a_{12}\bar z_1\bar z_2 + cz_1\bar z_2 + \bar c\bar z_1 z_2 + a_{22}z_2^2 + \bar a_{22}\bar z_2^2 \tag{4.1}$$

that system also admits the Liapunov function

$$L_1 = k|z_1|^2 + |z_2|^2 + cz_1\bar z_2 + \bar c\bar z_1 z_2 \tag{4.2}$$

where $k > 0$ is real, and a_{kj} and c are complex numbers.

Proof. If $z_1(t), z_2(t)$ is a solution of system (2.1), then $e^{i\alpha}z_1(t), e^{i\alpha}z_2(t)$ are also its solutions and, consequently, $L(e^{i\alpha}z_1, e^{i\alpha}z_2)$ is the Liapunov function of system (2.1). Then

$$L_1 = \frac{1}{2\pi}\int_0^{2\pi} L(e^{i\alpha}z_1, e^{i\alpha}z_2)\,d\alpha = k|z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}\,(cz_1\bar z_2)$$

is also the Liapunov function of that system.

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*) Shnol', E. E. and Khazin, L. G., On stability of stationary solutions of general systems of differential equations close to critical cases. Preprint No. 91 of the Inst. of Appl. Math., Akad. Nauk SSSR, 1979.

Let us determine the conditions for k and $c = c_1 + ic_2$ under which L_1 is a Liapunov function. Denoting $|c|^2 = c_0^2$ and assuming $c_0^2 < k$ we find that the quadratic form (4.2) is positive definite, i.e.

$$L_1 = (\sqrt{k}\rho_1 - c_0\sqrt{\rho_2 / k})^2 + \rho_2(1 - c_0^2/k) > 0, \ \rho_j = |z_j|^2$$

Let us stipulate $(dL_1 / dt)_{(2.1)} < 0$. Omitting intermediate calculations, we obtain

$$(dL_2 / dt)|_{(2.1)} = 2\rho_2^2 P_2(x); \quad x = \rho_1 / \rho_2 > 0 \tag{4.3}$$

$$P_2(x) = ax^2 + bx + d; \quad a = \alpha + {}^1/_2 \sqrt{C^2 + D^2}$$

$$b = \gamma + \sqrt{A^2 + B^2} + {}^1/_2(\sqrt{C^2 + D^2} + \sqrt{E^2 + F^2}), \quad d = \beta + {}^1/_2\sqrt{E^2 + F^2}$$

$$\alpha = ka_{11} + b_5(c_1\cos\psi_5 + c_2\sin\psi_5), \quad \beta = a_{22} + b_3(c_1\cos\psi_3 - c_2\sin\psi_3), \gamma = ka_{12} + a_{21} + b_1(c_1\cos\psi_1 - c_2\sin\psi_1) + b_6(c_1\cos\psi_6 + c_2\sin\psi_6), A = kb_4\cos\psi_4 + b_7\cos\psi_7 + c_1(b_2\cos\psi_2 + b_8\cos\psi_8) + c_2(b_8\sin\psi_8 - b_2\sin\psi_2)$$

$$B = -kb_4\sin\psi_4 + b_7\sin\psi_7 + c_1(b_2\sin\psi_2 + b_8\sin\psi_8) + c_2(b_2\cos\psi_2 + b_8\cos\psi_8)$$

$$C = k(b_1\cos\psi_1 + b_2\cos\psi_2) + b_5\cos\psi_5 + c_1(a_{11} + a_{21}) + c_2(c_{21} - c_{11}) + b_7(c_1\cos\psi_7 + c_2\sin\psi_7)$$

$$D = k(b_2\sin\psi_2 - b_1\sin\psi_1) + b_7(c_1\sin\psi_7 - c_2\cos\psi_7) + b_5\sin\psi_5 + c_1(c_{11} - c_{21}) + c_2(a_{11} + a_{21})$$

$$E = kb_3\cos\psi_3 + b_6\cos\psi_6 + b_8\cos\psi_8 + c_1(a_{12} + a_{22}) + c_2(c_{22} - c_{12}) - b_4(c_1\cos\psi_4 - c_2\sin\psi_4)$$

$$F = -kb_3\sin\psi_3 + b_6\sin\psi_6 - b_8\sin\psi_8 + c_1(c_{12} - c_{22}) + c_2(a_{12} + a_{22}) - b_4(c_1\sin\psi_4 + c_2\cos\psi_4)$$

if for $x > 0$, $P_2(x) < 0$, L_1 is a Liapunov function. The inequality $P_2(x) < 0 \ (x > 0)$ is satisfied under the following three conditions: 1) $a < 0$; 2) $d < 0$; and 3) either the roots of $P_2(x)$ are negative $(P_2^{\cdot}(0) = b < 0)$ or there no real roots $(b^2 - 4ad < 0)$.

Let us formulate the sufficient conditions for stability. Let

$$L_1 = k|z_1|^2 + |z_2|^2 + cz_1\bar{z}_2 + \bar{c}\bar{z}_1\bar{z}_2, \quad P_2(x) = a(k, c)x^2 + b(k, c)x + d(k, c)$$

and

$$I_1 = \{k, c : a > 0\}; \quad I_2 = \{k, c : d < 0\}, \quad I_3 = \{k, c : b < 0 \vee b^2 - 4ad < 0\}$$

Theorem. The equilibrium position $z = 0$ of system (2.1) is asymptotically stable, if $I_1 \cap (I_2 \cup I_3) \neq \varnothing$.

Examples of use of this theorem were given in Sect.2.

5. Limit situations. 1°. Small coefficients at resonance terms.

Theorem. If for fairly small $\varepsilon > 0$, $|B_j| < \varepsilon$, system (2.1) is asymptotically stable or unstable simultaneously with system (1.2).

Proof. a) If the asymptotic stability criterion for system (1.2) is satisfied, that system has a homogeneous Liapunov function L. It follows from the general theorems on homogeneous system stability such L is a Liapunov function also for system (2.1) when ε is fairly small /6/.

b) Let the equilibrium position of system (1.2) roughly unstable, i.e. when at least one of the stability criterion conditions is violated. Then the angle subsystem of system (3.2) when $b_j = 0$ has a nondegenerate stationary point (θ_0, ψ_0), i.e.

$$f = (\theta_0, \psi_0) = g(\theta_0, \psi_0) = 0; \quad d(f, g) / d(\theta, \psi)|_{\theta_0, \psi_0} \neq 0, \quad \Pi(\theta_0, \psi_0) > \lambda > 0$$

Hence, when ε is fairly small, the algebraic system $f(\theta, \psi) = g(\theta, \psi) = 0$ has the solution (θ_{10}, ψ_{10}) and $\Pi(\theta_{10}, \psi_{10}) > \lambda / 2 > 0$. The theorem in Sect.3 ensures under such conditions the instability of system (3.2) and of the complete resonance system.

2°. Large coefficients at resonance terms. Theorem. If $B_j = B_j'\varepsilon^{-1}$, then for fairly small ε system (2.1) is unstable.

Proof. We introduce in system (2.1) the new time $dt = \varepsilon dt_1$, and obtain

$$dz_1 / dt_1 = \varepsilon i\omega z_1 + \varepsilon z_1[A_{11}|z_1|^2 + A_{12}|z_2|^2] + B_1'|z_1|^2 z_2 + B_2' z_1^2 \bar{z}_2 + B_3' z_2|z_2|^2 + B_4'\bar{z}_1 z_2^2 \tag{5.1}$$

$$dz_2 / dt_1 = \varepsilon i\omega z_2 + \varepsilon z_2[A_{21}|z_1|^2 + A_{22}|z_2|^2] + B_5'|z_1|^2 z_1 + B_6' z_1|z_2|^2 + B_7' z_1^2 \bar{z}_2 + B_8'\bar{z}_1 z_2^2$$

Let us set $\varepsilon = 0$. Then the obtained system is roughly unstable, since the angle subsystem of system (3.2) (when $a_{ij} = 0$) has a nondegenerate stationary point (θ_0, ψ_0), i.e. $f(\theta_0, \psi_0) = g(\theta_0, \psi_0) = 0$ and $\Pi(\theta_0, \psi_0) > \lambda > 0$.

Because of this, for fairly small ε the algebraic system $f(\theta, \psi) = g(\theta, \psi) = 0$ has a steady solution (θ_{10}, ψ_{10}) and $\Pi(\theta_{10}, \psi_{10}) > \lambda / 2 > 0$. The theorem in Sect.3 ensures in this case the instability of system (5.1) and, by the same token, that of system (2.1).

REFERENCES

1. KAMENKOV G.V., Oscillation stability of nonlinear systems. Collected Works, Vol.2, Moscow, "Nauka", 1972.

2. MALKIN I.G., Theory of Motion Stability. Moscow, "Nauka", 1966.

3. MOLCHANOV A.M., Stability in the case of neutrality of linear approximation. Dokl. Akad. Nauk SSSR, Vol.141, No.1, 1961.

4. KUNITSYN A.L., On stability in the critical case of pure imaginary roots in the presence of inner resonance. Differents. Uravneniia, Vol.7, No.9, 1971.

5. KHAZINA G.G., Certain stability questions in the presence of resonances. PMM, Vol.38, No.1, 1974.

6. KRASOVSKII N.N., On stability to the first approximation, PMM, Vol.19, No.5, 1955.

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